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ON THE COHEN-MACAULAY PROPERTY OF MULTIPLICATIVE INVARIANTS

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Abstract. We investigate the Cohen-Macaulay property for rings of invariants under multiplicative actions of a finite group $\mathcal{G}$. By definition, these are $\mathcal{G}$-actions on Laurent polynomial algebras $\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ that stabilize the multiplicative group consisting of all monomials in the variables x_i. For the most part, we concentrate on the case where the base ring $\Bbbk$ is $\mathbb{Z}$. Our main result states that if $\mathcal{G}$ acts non-trivially and the invariant ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\mathcal{G}}$ is Cohen-Macaulay, then the abelianized isotropy groups $\mathcal{G}_m^{\mathrm{ab}}$ of all monomials m are generated by the bireflections in $\mathcal{G}_m$ and at least one $\mathcal{G}_m^{\mathrm{ab}}$ is non-trivial. As an application, we prove the multiplicative version of Kemper's 3-copies conjecture.

Introduction

This article is a sequel to [LPk]. Unlike in [LPk], however, our focus will be specifically on multiplicative invariants. In detail, let $L \cong \mathbb{Z}^n$ denote a lattice on which a finite group $\mathcal{G}$ acts by automorphisms. The $\mathcal{G}$-action on L extends uniquely to an action by $\Bbbk$-algebra automorphisms on the group algebra $\Bbbk[L] \cong \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ over any commutative base ring $\Bbbk$. We are interested in the question of when the subalgebra $\Bbbk[L]^{\mathcal{G}}$ consisting of all $\mathcal{G}$-invariant elements of $\Bbbk[L]$ has the Cohen-Macaulay property. The reader is assumed to have some familiarity with Cohen-Macaulay rings; a good reference on this subject is [BH].

It is a standard fact that $\Bbbk[L]$ is Cohen-Macaulay precisely if $\Bbbk$ is. On the other hand, while $\Bbbk[L]^{\mathcal{G}}$ can only be Cohen-Macaulay when $\Bbbk$ is so, the latter condition is far from sufficient, and rather stringent additional conditions on the action of $\mathcal{G}$ on L are required to ensure that $\Bbbk[L]^{\mathcal{G}}$ is Cohen-Macaulay. Remarkably, the question of whether or not $\Bbbk[L]^{\mathcal{G}}$ is Cohen-Macaulay, for any given base ring $\Bbbk$, depends only on the rational isomorphism class of the lattice L, that is, the isomorphism class of $L \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}[\mathcal{G}]$-module; see Proposition 3.4 below. This is in striking contrast with most other ring theoretic properties of $\Bbbk[L]^{\mathcal{G}}$ (e.g., regularity, structure of the class group) which tend to be sensitive to the $\mathbb{Z}$-type of L. For an overview, see [L$_1$].

We will largely concentrate on the case where the base ring $\Bbbk$ is $\mathbb{Z}$. This is justified in part by the fact that if $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay, then likewise is $\Bbbk[L]^{\mathcal{G}}$

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for any Cohen-Macaulay base ring $\Bbbk$ (Lemma 3.2). Assuming $\mathbb{Z}[L]^{\mathcal{G}}$ to be Cohen-Macaulay, we aim to derive group theoretical consequences for the isotropy groups $\mathcal{G}_m = \{g \in \mathcal{G} \mid g(m) = m\}$ with $m \in L$. An element $g \in \mathcal{G}$ will be called a *k-reflection* on L if the sublattice $[g, L] = \{g(m) - m \mid m \in L\}$ of L has rank at most k or, equivalently, if the g-fixed points of the $\mathbb{Q}$-space $L \otimes_{\mathbb{Z}} \mathbb{Q}$ have codimension at most k. As usual, k-reflections with $k = 1$ and $k = 2$ will be referred to as *reflections* and *bireflections*. For any subgroup $\mathcal{H} \leq \mathcal{G}$, we let $\mathcal{H}^{(2)}$ denote the subgroup generated by the elements of $\mathcal{H}$ that act as bireflections on L. Our main result now reads as follows.

Theorem. *Assume that $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay. Then $\mathcal{G}_m/\mathcal{G}_m^{(2)}$ is a perfect group (i.e., equal to its commutator subgroup) for all $m \in L$. If $\mathcal{G}$ acts non-trivially on L, then some $\mathcal{G}_m$ is non-perfect.*

It would be interesting to determine if the conclusion of the theorem can be strengthened to the effect that all isotropy groups $\mathcal{G}_m$ are in fact generated by bireflections on L. I do not know if, for the latter to occur, it is sufficient that $\mathcal{G}$ is generated by bireflections. The corresponding fact for reflection groups is known to be true: if $\mathcal{G}$ is generated by reflections on L (or, equivalently, on $L \otimes_{\mathbb{Z}} \mathbb{Q}$), then so are all isotropy groups $\mathcal{G}_m$; see [St, Theorem 1.5] or [Bou_1, Exercise 8(a) on p. 139].

There is essentially a complete classification of finite linear groups generated by bireflections. In arbitrary characteristic, this is due to Guralnick and Saxl [GuS]; for the case of characteristic zero, see Huffman and Wales [HuW]. Bireflection groups have been of interest in connection with the problem of determining all finite linear groups whose algebra of polynomial invariants is a complete intersection. Specifically, suppose that $\mathcal{G} \leq \mathrm{GL}(V)$ for some finite-dimensional vector space V and let $\mathcal{O}(V) = \mathsf{S}(V^*)$ denote the algebra of polynomial functions on V. It was shown by Kac and Watanabe [KW] and independently by Gordeev [G_1] that if the algebra $\mathcal{O}(V)^{\mathcal{G}}$ of all $\mathcal{G}$-invariant polynomial functions is a complete intersection, then $\mathcal{G}$ is generated by bireflections on V. The classification of all groups $\mathcal{G}$ so that $\mathcal{O}(V)^{\mathcal{G}}$ is a complete intersection has been achieved by Gordeev [G_2] and by Nakajima [N].

The last assertion of the above Theorem implies in particular that if $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay and $\mathcal{G}$ acts non-trivially on L, then some element of $\mathcal{G}$ acts as a non-trivial bireflection on L. Hence we obtain the following multiplicative version of Kemper's 3-copies conjecture:

Corollary. *If $\mathcal{G}$ acts non-trivially on L and $r \geq 3$, then $\mathbb{Z}[L^{\oplus r}]^{\mathcal{G}}$ is not Cohen-Macaulay.*

The 3-copies conjecture was formulated by Kemper [K_1, Vermutung 3.12] in the context of polynomial invariants. Using the above notation, the original conjecture states that if $1 \neq \mathcal{G} \leq \mathrm{GL}(V)$ and the characteristic of the base field of V divides the order of $\mathcal{G}$ ("modular case"), then the invariant algebra $\mathcal{O}(V^{\oplus r})^{\mathcal{G}}$ will not be Cohen-Macaulay for any $r \geq 3$. This is still open. The main factors contributing to our success in the multiplicative case are the following:

- Multiplicative actions are permutation actions: $\mathcal{G}$ permutes the $\Bbbk$-basis of $\Bbbk[L]$ consisting of all "monomials", corresponding to the elements of the lattice L. Consequently, the cohomology $H^*(\mathcal{G}, \Bbbk[L])$ is simply the direct sum of the various $H^*(\mathcal{G}_m, \Bbbk)$ with m running over a transversal for the $\mathcal{G}$-orbits in L.

- Up to conjugacy, there are only finitely many finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$, and these groups are explicitly known for small n. A crucial observation for our purposes is the following: if $\mathcal{G}$ is a non-trivial finite perfect subgroup of $\mathrm{GL}_n(\mathbb{Z})$ such that no $1 \neq g \in \mathcal{G}$ has eigenvalue 1, then $\mathcal{G}$ is isomorphic to the binary icosahedral group and $n \geq 8$; see Lemma 2.3 below.

A brief outline of the contents of the this article is as follows. The short preliminary Section 1 is devoted to general actions of a finite group $\mathcal{G}$ on a commutative ring R. This material relies rather heavily on [LPk]. We liberate a technical result from [LPk] from any a priori hypotheses on the characteristic; the new version (Proposition 1.4) states that if R and $R^{\mathcal{G}}$ are both Cohen-Macaulay and $H^i(\mathcal{G}, R) = 0$ for $0 < i < k$, then $H^k(\mathcal{G}, R)$ is detected by $k+1$-reflections. Section 2 then specializes to the case of multiplicative actions. We assemble the main tools required for the proof of the Theorem, which is presented in Section 3. The article concludes with a brief discussion of possible avenues for further investigation and some examples.

1. Finite group actions on rings

1.1. In this section, R will be a commutative ring on which a finite group $\mathcal{G}$ acts by ring automorphisms $r \mapsto g(r)$ $(r \in R, g \in \mathcal{G})$. The subring of $\mathcal{G}$-invariant elements of R will be denoted by $R^{\mathcal{G}}$.

1.2. **Generalized reflections.** Following [GK], we will say an element $g \in \mathcal{G}$ acts as a *k-reflection* on R if g belongs to the inertia group

$$I_{\mathcal{G}}(\mathfrak{P}) = \{g \in \mathcal{G} \mid g(r) - r \in \mathfrak{P} \ \forall r \in R\}$$

of some prime ideal $\mathfrak{P} \in \operatorname{Spec} R$ with height $\mathfrak{P} \leq k$. The cases $k = 1$ and $k = 2$ will be referred to as *reflections* and *bireflections*, respectively. Define the ideal $I_R(g)$ of R by

$$I_R(g) = \sum_{r \in R} (g(r) - r)R \ .$$

Evidently, $\mathfrak{P} \supseteq I_R(g)$ is equivalent to $g \in I_{\mathcal{G}}(\mathfrak{P})$. Thus:

$$g \text{ is a } k\text{-reflection on } R \text{ if and only if height } I_R(g) \leq k.$$

For each subgroup $\mathcal{H} \leq \mathcal{G}$, we put

$$I_R(\mathcal{H}) = \sum_{g \in \mathcal{H}} I_R(g) \ .$$

It suffices to let g run over a set of generators of the group $\mathcal{H}$ in this sum.

1.3. **A height estimate.** The cohomology $H^*(\mathcal{G}, R) = \bigoplus_{n \geq 0} H^n(\mathcal{G}, R)$ has a canonical $R^{\mathcal{G}}$-module structure: for each $r \in R^{\mathcal{G}}$, the map $\rho\colon R \to R$, $s \mapsto rs$, is $\mathcal{G}$-equivariant and hence it induces a map on cohomology $\rho_* : H^*(\mathcal{G}, R) \to H^*(\mathcal{G}, R)$. The element r acts on $H^*(\mathcal{G}, R)$ via ρ_*. Let $\operatorname{res}^{\mathcal{G}}_{\mathcal{H}}\colon H^*(\mathcal{G}, R) \to H^*(\mathcal{H}, R)$ denote the restriction map.

The following lemma extends [LPk, Proposition 1.4].

Lemma. *For any $x \in H^*(\mathcal{G}, R)$,*

$$\text{height} \operatorname{ann}_{R^{\mathcal{G}}}(x) \geq \inf\{\text{height } I_R(\mathcal{H}) \mid \mathcal{H} \leq \mathcal{G}, \operatorname{res}^{\mathcal{G}}_{\mathcal{H}}(x) \neq 0\} \ .$$

Proof. Put $\mathfrak{X} = \{\mathcal{H} \leq \mathcal{G} \mid \mathrm{res}^{\mathcal{G}}_{\mathcal{H}}(x) = 0\}$. For each $\mathcal{H} \leq \mathcal{G}$, let $R^{\mathcal{G}}_{\mathcal{H}}$ denote the image of the relative trace map $R^{\mathcal{H}} \to R^{\mathcal{G}}$, $r \mapsto \sum_g g(r)$, where g runs over a transversal for the cosets $g\mathcal{H}$ of $\mathcal{H}$ in $\mathcal{G}$. By [LPk, Lemma 1.3],

$$R^{\mathcal{G}}_{\mathcal{H}} \subseteq \mathrm{ann}_{R^{\mathcal{G}}}(x) \quad \text{for all } \mathcal{H} \in \mathfrak{X}.$$

To prove the lemma, we may assume that $\mathrm{ann}_{R^{\mathcal{G}}}(x)$ is a proper ideal of $R^{\mathcal{G}}$; for, otherwise height $\mathrm{ann}_{R^{\mathcal{G}}}(x) = \infty$. Choose a prime ideal $\mathfrak{p}$ of $R^{\mathcal{G}}$ with $\mathfrak{p} \supseteq \mathrm{ann}_{R^{\mathcal{G}}}(x)$ and height $\mathfrak{p}$ = height $\mathrm{ann}_{R^{\mathcal{G}}}(x)$. If $\mathfrak{P}$ is a prime of R that lies over $\mathfrak{p}$, then

$$R^{\mathcal{G}}_{\mathcal{H}} \subseteq \mathfrak{P} \quad \text{for all } \mathcal{H} \in \mathfrak{X}$$

and height $\mathfrak{P}$ = height $\mathfrak{p}$. By [LPk, Lemma 1.1], the above inclusion implies that

$$[I_{\mathcal{G}}(\mathfrak{P}) : I_{\mathcal{H}}(\mathfrak{P})] \in \mathfrak{P} \quad \text{for all } \mathcal{H} \in \mathfrak{X}.$$

Put $p = \mathrm{char}\, R/\mathfrak{P}$ and let $\mathcal{P} \leq I_{\mathcal{G}}(\mathfrak{P})$ be a Sylow p-subgroup of $I_{\mathcal{G}}(\mathfrak{P})$ (so $\mathcal{P} = 1$ if $p = 0$). Then $I_R(\mathcal{P}) \subseteq \mathfrak{P}$ and $[I_{\mathcal{G}}(\mathfrak{P}) : \mathcal{P}] \notin \mathfrak{P}$. Hence, $\mathcal{P} \notin \mathfrak{X}$ and height $I_R(\mathcal{P}) \leq$ height $\mathfrak{P}$ = height $\mathrm{ann}_{R^{\mathcal{G}}}(x)$. This proves the lemma. □

We remark that the lemma and its proof carry over verbatim to the more general situation where $H^*(\mathcal{G}, R)$ is replaced by $H^*(\mathcal{G}, M)$, where M is some module over the skew group ring of $\mathcal{G}$ over R; cf. [LPk]. However, we will not be concerned with this generalization here.

1.4. **A necessary condition.** In this section, we assume that R is noetherian as an $R^{\mathcal{G}}$-module. This assumption is satisfied whenever R is an affine algebra over some noetherian subring $\Bbbk \subseteq R^{\mathcal{G}}$; see [Bou$_2$, Théorème 2 on p. 33]. Put

$$\mathfrak{X}_k = \{\mathcal{H} \leq \mathcal{G} \mid \text{height } I_R(\mathcal{H}) \leq k\} \ . \tag{1.1}$$

Note that each $\mathcal{H} \in \mathfrak{X}_k$ consists of k-reflections on R. The following proposition is a characteristic-free version of [LPk, Proposition 4.1].

Proposition. *Assume that R and $R^{\mathcal{G}}$ are Cohen-Macaulay. If $H^i(\mathcal{G}, R) = 0$ for $0 < i < k$, then the restriction map*

$$\mathrm{res}^{\mathcal{G}}_{\mathfrak{X}_{k+1}} : H^k(\mathcal{G}, R) \to \prod_{\mathcal{H} \in \mathfrak{X}_{k+1}} H^k(\mathcal{H}, R)$$

is injective.

Proof. We may assume that $H^k(\mathcal{G}, R) \neq 0$. Let $x \in H^k(\mathcal{G}, R)$ be non-zero and put $\mathfrak{a} = \mathrm{ann}_{R^{\mathcal{G}}}(x)$. By [LPk, Proposition 3.3], depth $\mathfrak{a} \leq k + 1$. Since $R^{\mathcal{G}}$ is Cohen-Macaulay, depth $\mathfrak{a}$ = height $\mathfrak{a}$. Thus, Lemma 1.3 implies that $k + 1 \geq$ height $I_R(\mathcal{H})$ for some $\mathcal{H} \leq \mathcal{G}$ with $\mathrm{res}^{\mathcal{G}}_{\mathcal{H}}(x) \neq 0$. The Proposition follows. □

Note that the vanishing hypothesis on $H^i(\mathcal{G}, R)$ is vacuous for $k = 1$. Thus, $H^1(\mathcal{G}, R)$ is detected by bireflections whenever R and $R^{\mathcal{G}}$ are both Cohen-Macaulay.

2. Multiplicative actions

2.1. For the remainder of this article, L will denote a lattice on which the finite group $\mathcal{G}$ acts by automorphisms $m \mapsto g(m)$ ($m \in L, g \in \mathcal{G}$). The group algebra of L over some commutative base ring $\Bbbk$ will be denoted by $\Bbbk[L]$. We will use additive notation in L. The $\Bbbk$-basis element of $\Bbbk[L]$ corresponding to the lattice element $m \in L$ will be written as

$$\mathbf{x}^m \ ;$$

so $\mathbf{x}^0 = 1$, $\mathbf{x}^{m+m'} = \mathbf{x}^m\mathbf{x}^{m'}$, and $\mathbf{x}^{-m} = (\mathbf{x}^m)^{-1}$. The action of $\mathcal{G}$ on L extends uniquely to an action by $\Bbbk$-algebra automorphisms on $\Bbbk[L]$:

$$g(\sum_{m\in L} k_m \mathbf{x}^m) = \sum_{m\in L} k_m \mathbf{x}^{g(m)} \ .$$

The invariant algebra $\Bbbk[L]^{\mathcal{G}}$ is a free $\Bbbk$-module: a $\Bbbk$-basis is given by the $\mathcal{G}$-orbit sums $\sigma(m) = \sum_{m'\in\mathcal{G}(m)} \mathbf{x}^{m'}$, where $\mathcal{G}(m)$ denotes the $\mathcal{G}$-orbit of $m \in L$. Since all orbit sums are defined over $\mathbb{Z}$, we have

$$\Bbbk[L]^{\mathcal{G}} = \Bbbk \otimes_{\mathbb{Z}} \mathbb{Z}[L]^{\mathcal{G}} \ . \tag{2.1}$$

2.2. Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$. We compute the height of the ideal $I_{\Bbbk[L]}(\mathcal{H})$ from §1.2. Let

$$L^{\mathcal{H}} = \{m \in L \mid g(m) = m \text{ for all } g \in \mathcal{H}\}$$

denote the lattice of $\mathcal{H}$-invariants in L and define the sublattice $[\mathcal{H}, L]$ of L by

$$[\mathcal{H}, L] = \sum_{g\in\mathcal{H}} [g, L] \ ,$$

where $[g, L] = \{g(m) - m \mid m \in L\}$. It suffices to let g run over a set of generators of the group $\mathcal{H}$ in the above formulas.

Lemma. *With the above notation,* $\Bbbk[L]/I_{\Bbbk[L]}(\mathcal{H}) \cong \Bbbk[L/[\mathcal{H}, L]]$ *and*

$$\operatorname{height} I_{\Bbbk[L]}(\mathcal{H}) = \operatorname{rank}[\mathcal{H}, L] = \operatorname{rank} L - \operatorname{rank} L^{\mathcal{H}} \ .$$

Proof. Since the ideal $I_{\Bbbk[L]}(\mathcal{H})$ is generated by the elements $\mathbf{x}^{g(m)-m}-1$ with $m \in L$ and $g \in \mathcal{H}$, the isomorphism $\Bbbk[L]/I_{\Bbbk[L]}(\mathcal{H}) \cong \Bbbk[L/[\mathcal{H}, L]]$ is clear.

To prove the equality $\operatorname{rank}[\mathcal{H}, L] = \operatorname{rank} L - \operatorname{rank} L^{\mathcal{H}}$, note that the rational group algebra $\mathbb{Q}[\mathcal{H}]$ is the direct sum of the ideals $\mathbb{Q}\left(\sum_{g\in\mathcal{H}} g\right)$ and $\sum_{g\in\mathcal{H}} \mathbb{Q}(g-1)$. This implies $L \otimes_{\mathbb{Z}} \mathbb{Q} = \left(L^{\mathcal{H}} \otimes_{\mathbb{Z}} \mathbb{Q}\right) \oplus ([\mathcal{H}, L] \otimes_{\mathbb{Z}} \mathbb{Q})$. Hence, $\operatorname{rank} L = \operatorname{rank} L^{\mathcal{H}} + \operatorname{rank}[\mathcal{H}, L]$.

To complete the proof, it suffices to show that

$$\operatorname{height} \mathfrak{P} = \operatorname{rank}[\mathcal{H}, L]$$

holds for any minimal covering prime $\mathfrak{P}$ of $I_{\Bbbk[L]}(\mathcal{H})$. Put $A = L/[\mathcal{H}, L]$ and $\bar{\mathfrak{P}} = \mathfrak{P}/I_{\Bbbk[L]}(\mathcal{H})$, a minimal prime of $\Bbbk[L]/I_{\Bbbk[L]}(\mathcal{H}) = \Bbbk[A]$. Further, put $\mathfrak{p} = \bar{\mathfrak{P}} \cap \Bbbk = \mathfrak{P} \cap \Bbbk$. Since the extension $\Bbbk \hookrightarrow \Bbbk[A] = \Bbbk[L]/I_{\Bbbk[L]}(\mathcal{H})$ is free, $\mathfrak{p}$ is a minimal prime of $\Bbbk$; see [Bou$_3$, Cor. on p. AC VIII.15]. Hence, descending chains of primes in $\Bbbk[L]$ starting with $\mathfrak{P}$ correspond in a 1-to-1 fashion to descending chains of primes of $Q(\Bbbk/\mathfrak{p})[L]$ starting with the prime that is generated by $\mathfrak{P}$. Thus, replacing $\Bbbk$ by $Q(\Bbbk/\mathfrak{p})$, we may assume that $\Bbbk$ is a field. But then

$$\operatorname{height} \mathfrak{P} = \dim \Bbbk[L] - \dim \Bbbk[L]/\mathfrak{P} = \operatorname{rank} L - \dim \Bbbk[L]/\mathfrak{P} \ .$$

Let $\bar{\mathfrak{P}}_0 = \bar{\mathfrak{P}} \cap \Bbbk[A_0]$, where A_0 denotes the torsion subgroup of A. Since $\bar{\mathfrak{P}}$ is minimal, we have $\bar{\mathfrak{P}} = \bar{\mathfrak{P}}_0\Bbbk[A]$ and so $\Bbbk[L]/\mathfrak{P} \cong \Bbbk_0[A/A_0]$, where $\Bbbk_0 = \Bbbk[A_0]/\bar{\mathfrak{P}}_0$ is a field. Thus, $\dim \Bbbk[L]/\mathfrak{P} = \operatorname{rank} A/A_0$. Finally, $\operatorname{rank} A/A_0 = \operatorname{rank} A = \operatorname{rank} L - \operatorname{rank}[\mathcal{H}, L]$, which completes the proof. □

Specializing the lemma to the case where $\mathcal{H} = \langle g\rangle$ for some $g \in \mathcal{G}$, we see that g acts as a k-reflection on $\Bbbk[L]$ if and only if g acts as a k-reflection on L, that is,

$$\operatorname{rank}[g, L] \leq k \ .$$

Moreover, the collection of subgroups $\mathfrak{X}_k$ in equation (1.1) can now be written as

$$\mathfrak{X}_k = \{\mathcal{H} \leq \mathcal{G} \mid \operatorname{rank} L/L^{\mathcal{H}} \leq k\} \ . \tag{2.2}$$

2.3. **Fixed-point-free lattices for perfect groups.** The $\mathcal{G}$-action on L is called *fixed-point-free* if $g(m) \neq m$ holds for all $0 \neq m \in L$ and $1 \neq g \in \mathcal{G}$. Recall also that the group $\mathcal{G}$ is said to be *perfect* if $\mathcal{G}^{\mathrm{ab}} = \mathcal{G}/[\mathcal{G}, \mathcal{G}] = 1$.

Lemma. *Assume that $\mathcal{G}$ is a non-trivial perfect group acting fixed-point-freely on the non-zero lattice L. Then $\mathcal{G}$ is isomorphic to the binary icosahedral group $2.\mathcal{A}_5 \cong \mathrm{SL}_2(\mathbb{F}_5)$ and $\operatorname{rank} L$ is a multiple of 8.*

Proof. Put $V = L \otimes_{\mathbb{Z}} \mathbb{C}$, a non-zero fixed-point-free $\mathbb{C}[\mathcal{G}]$-module. By a well-known theorem of Zassenhaus (see [Wo, Theorem 6.2.1]), $\mathcal{G}$ is isomorphic to the binary icosahedral group $2.\mathcal{A}_5$ and the irreducible constituents of V are 2-dimensional. The binary icosahedral group has two irreducible complex representations of degree 2; they are Galois conjugates of each other and both have Frobenius-Schur indicator -1. We denote the corresponding $\mathbb{C}[\mathcal{G}]$-modules by V_1 and V_2. Both V_i occur with the same multiplicity in V, since V is defined over $\mathbb{Q}$. Thus, $V \cong (V_1 \oplus V_2)^m$ for some m and $\operatorname{rank} L = 4m$. We have to show that m is even. Since both V_i have indicator -1, it follows that $V_1 \oplus V_2$ is not defined over $\mathbb{R}$, whereas each V_i^2 is defined over $\mathbb{R}$; see [I, (9.21)]. Thus, $V_1 \oplus V_2$ represents an element x of order 2 in the cokernel of the scalar extension map $G_0(\mathbb{R}[\mathcal{G}]) \to G_0(\mathbb{C}[\mathcal{G}])$, and $mx = 0$. Therefore, m must be even, as desired. □

We remark that the binary icosahedral group $2.\mathcal{A}_5$ is isomorphic to the subgroup of the non-zero quaternions $\mathbb{H}^*$ that is generated by $(a+i+ja^*)/2$ and $(a+j+ka^*)/2$, where $a = (1+\sqrt{5})/2$ and $a^* = (1-\sqrt{5})/2$ and $\{1, i, j, k\}$ is the standard $\mathbb{R}$-basis of $\mathbb{H}$. Thus, letting $2.\mathcal{A}_5$ act on $\mathbb{H}$ via left multiplication, $\mathbb{H}$ becomes a 2-dimensional fixed-point-free complex representation of $2.\mathcal{A}_5$. It is easy to see that this representation can be realized over $K = \mathbb{Q}(i, \sqrt{5})$; so $\mathbb{H} = V \otimes_K \mathbb{C}$ with $\dim_{\mathbb{Q}} V = 2[K : \mathbb{Q}] = 8$. Any $2.\mathcal{A}_5$-lattice for V will be fixed-point-free and have rank 8.

2.4. **Isotropy groups.** The isotropy group of an element $m \in L$ in $\mathcal{G}$ will be denoted by $\mathcal{G}_m$; so

$$\mathcal{G}_m = \{g \in \mathcal{G} \mid g(m) = m\} \ .$$

The $\mathcal{G}$-lattice L is called *faithful* if $\operatorname{Ker}_{\mathcal{G}}(L) = \bigcap_{m \in L} \mathcal{G}_m = 1$. The following lemma, at least part (a), is well known. We include the proof for the reader's convenience.

Lemma. (a) *The set of isotropy groups $\{\mathcal{G}_m \mid m \in L\}$ is closed under conjugation and under taking intersections.*

(b) *Assume that the $\mathcal{G}$-lattice L is faithful. If $\mathcal{G}_m$ $(m \in L)$ is a minimal non-identity isotropy group, then $\mathcal{G}_m$ acts fixed-point-freely on $L/L^{\mathcal{G}_m} \neq 0$.*

Proof. Consider the $\mathbb{Q}[\mathcal{G}]$-module $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. The collection of isotropy groups $\mathcal{G}_m$ remains unchanged when allowing $m \in V$. Moreover, for any subgroup $\mathcal{H} \leq \mathcal{G}$, $L/L^{\mathcal{H}}$ is an $\mathcal{H}$-lattice with $L/L^{\mathcal{H}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong V/V^{\mathcal{H}}$.

(a) The first assertion is clear, since ${}^g\mathcal{G}_m = \mathcal{G}_{g(m)}$ holds for all $g \in \mathcal{G}, m \in V$. For the second assertion, let M be a non-empty subset of V and put $\mathcal{G}_M = \bigcap_{m \in M} \mathcal{G}_m$. We must show that $\mathcal{G}_M = \mathcal{G}_m$ for some $m \in V$. Put $W = V^{\mathcal{G}_M}$. If $g \in \mathcal{G} \setminus \mathcal{G}_M$, then $W^g = \{w \in W \mid g(w) = w\}$ is a proper subspace of W, since some element of M does not belong to W^g. Any $m \in W \setminus \bigcup_{g \in \mathcal{G} \setminus \mathcal{G}_M} W^g$ satisfies $\mathcal{G}_m = \mathcal{G}_M$.

(b) Let $\mathcal{H} = \mathcal{G}_m$ be a minimal non-identity member of $\{\mathcal{G}_m \mid m \in V\}$. As $\mathbb{Q}[\mathcal{H}]$-modules, we may identify V and $V^{\mathcal{H}} \oplus V/V^{\mathcal{H}}$. If $0 \neq v \in V/V^{\mathcal{H}}$, then $\mathcal{H}_v = \mathcal{H} \cap \mathcal{G}_v \subsetneq \mathcal{H}$. In view of (a), our minimality assumption on $\mathcal{H}$ forces $\mathcal{H}_v = 1$. Thus, $\mathcal{H}$ acts fixed-point-freely on $V/V^{\mathcal{H}}$, and hence on $L/L^{\mathcal{H}}$. □

Proposition. *Assume that L is a faithful $\mathcal{G}$-lattice such that all minimal isotropy groups $1 \neq \mathcal{G}_m$ $(m \in L)$ are perfect. Then* $\operatorname{rank} L/L^{\mathcal{H}} \geq 8$ *holds for every nonidentity subgroup $\mathcal{H} \leq \mathcal{G}$.*

In the notation of equation (2.2), the conclusion of the proposition can be stated as follows:

$$\mathfrak{X}_k = \{1\} \text{ for all } k < 8.$$

Proof of the Proposition. Put $\bar{\mathcal{H}} = \bigcap_{m \in L^{\mathcal{H}}} \mathcal{G}_m$. Then $\bar{H} \supseteq \mathcal{H}$ and $L^{\bar{\mathcal{H}}} = L^{\mathcal{H}}$. Lemma 2.4(a) further implies that $\bar{\mathcal{H}} = \mathcal{G}_m$ for some m. Replacing $\mathcal{H}$ by $\bar{H}$, we may assume that $\mathcal{H}$ is a nonidentity isotropy group. If $\mathcal{H}$ is not minimal then replace $\mathcal{H}$ by a smaller nonidentity isotropy group; this does not increase the value of $\operatorname{rank} L/L^{\mathcal{H}}$. Thus, we may assume that $\mathcal{H}$ is a minimal nonidentity isotropy group, and hence $\mathcal{H}$ is perfect. By Lemma 2.4(b), $\mathcal{H}$ acts fixed-point-freely on $L/L^{\mathcal{H}} \neq 0$ and Lemma 2.3 implies that $\operatorname{rank} L/L^{\mathcal{H}} \geq 8$, proving the proposition. □

2.5. **Cohomology.** Let $\mathfrak{X}$ denote any collection of subgroups of $\mathcal{G}$ that is closed under conjugation and under taking subgroups. We will investigate injectivity of the restriction map

$$\operatorname{res}^{\mathcal{G}}_{\mathfrak{X}} \colon H^k(\mathcal{G}, \Bbbk[L]) \to \prod_{\mathcal{H} \in \mathfrak{X}} H^k(\mathcal{H}, \Bbbk[L]) \ .$$

This map was considered in Proposition 1.4 for $\mathfrak{X} = \mathfrak{X}_{k+1}$.

Lemma. *The map* $\operatorname{res}^{\mathcal{G}}_{\mathfrak{X}} \colon H^k(\mathcal{G}, \Bbbk[L]) \to \prod_{\mathcal{H} \in \mathfrak{X}} H^k(\mathcal{H}, \Bbbk[L])$ *is injective if and only if the restriction maps*

$$H^k(\mathcal{G}_m, \Bbbk) \to \prod_{\substack{\mathcal{H} \in \mathfrak{X} \\ \mathcal{H} \leq \mathcal{G}_m}} H^k(\mathcal{H}, \Bbbk)$$

are injective for all $m \in L$.

Proof. As $\Bbbk[\mathcal{G}]$-module, $\Bbbk[L]$ is a permutation module:

$$\Bbbk[L] \cong \bigoplus_{m \in \mathcal{G} \setminus L} \Bbbk[\mathcal{G}/\mathcal{G}_m] \ ,$$

where $\Bbbk[\mathcal{G}/\mathcal{G}_m] = \Bbbk[\mathcal{G}] \otimes_{\Bbbk[\mathcal{G}_m]} \Bbbk$ and $\mathcal{G} \backslash L$ is a transversal for the $\mathcal{G}$-orbits in L. For each subgroup $\mathcal{H} \leq \mathcal{G}$,

$$\Bbbk[\mathcal{G}/\mathcal{G}_m]\big|_{\mathcal{H}} \cong \bigoplus_{g \in \mathcal{H} \backslash \mathcal{G} / \mathcal{G}_m} \Bbbk[\mathcal{H}/{}^g\mathcal{G}_m \cap \mathcal{H}] \ ;$$

see [CR, 10.13]. Therefore, $\operatorname{res}^{\mathcal{G}}_{\mathcal{H}}$ is the direct sum of the restriction maps

$$H^k(\mathcal{G}, \Bbbk[\mathcal{G}/\mathcal{G}_m]) \to H^k(\mathcal{H}, \Bbbk[\mathcal{G}/\mathcal{G}_m]) = \bigoplus_{g \in \mathcal{H} \backslash \mathcal{G} / \mathcal{G}_m} H^k(\mathcal{H}, \Bbbk[\mathcal{H}/{}^g\mathcal{G}_m \cap \mathcal{H}]) \ .$$

By the Eckmann-Shapiro Lemma [Br, III(5.2),(6.2)], $H^k(\mathcal{G}, \Bbbk[\mathcal{G}/\mathcal{G}_m]) \cong H^k(\mathcal{G}_m, \Bbbk)$ and $H^k(\mathcal{H}, \Bbbk[\mathcal{H}/{}^g\mathcal{G}_m \cap \mathcal{H}]) \cong H^k({}^g\mathcal{G}_m \cap \mathcal{H}, \Bbbk)$. In terms of these isomorphisms, the above restriction map becomes

$$\begin{array}{rcl} \rho_{\mathcal{H},m} \colon H^k(\mathcal{G}_m, \Bbbk) & \to & \bigoplus_{g \in \mathcal{H}\backslash\mathcal{G}/\mathcal{G}_m} H^k({}^g\mathcal{G}_m \cap \mathcal{H}, \Bbbk) \\ {[f]} & \mapsto & ([\underline{h} \mapsto f(g^{-1}\underline{h}g)])_g \end{array}$$

where $[\,.\,]$ denotes the cohomology class of a k-cocycle and $\underline{h}$ stands for a k-tuple of elements of ${}^g\mathcal{G}_m \cap \mathcal{H}$. Therefore,

$$\operatorname{Ker} \rho_{\mathcal{H},m} = \bigcap_{g \in \mathcal{H}\backslash\mathcal{G}/\mathcal{G}_m} \operatorname{Ker}\left(\operatorname{res}^{\mathcal{G}_m}_{\mathcal{G}_m \cap \mathcal{H}^g} \colon H^k(\mathcal{G}_m, \Bbbk) \to H^k(\mathcal{G}_m \cap \mathcal{H}^g, \Bbbk)\right) .$$

Thus, $\operatorname{Ker} \operatorname{res}^{\mathcal{G}}_{\mathfrak{X}}$ is isomorphic to the direct sum of the kernels of the restriction maps

$$H^k(\mathcal{G}_m, \Bbbk) \to \prod_{\mathcal{H} \in \mathfrak{X}} H^k(\mathcal{G}_m \cap \mathcal{H}^g, \Bbbk)$$

with $m \in \mathcal{G}\backslash L$. Finally, by hypothesis on $\mathfrak{X}$, the groups $\mathcal{G}_m \cap \mathcal{H}^g$ with $\mathcal{H} \in \mathfrak{X}$ are exactly the groups $\mathcal{H} \in \mathfrak{X}$ with $\mathcal{H} \leq \mathcal{G}_m$. The lemma follows. □

Corollary. *Let $\Bbbk = \mathbb{Z}/(|\mathcal{G}|)$ and $k = 1$. Then $\operatorname{res}^{\mathcal{G}}_{\mathfrak{X}}$ injective if and only if all $\mathcal{G}_m^{\mathrm{ab}}$ ($m \in L$) are generated by the images of the subgroups $\mathcal{H} \leq \mathcal{G}_m$ with $\mathcal{H} \in \mathfrak{X}$.*

Proof. By the lemma with $k = 1$, the hypothesis on the restriction map says that all restrictions

$$H^1(\mathcal{G}_m, \Bbbk) \to \prod_{\substack{\mathcal{H} \in \mathfrak{X} \\ \mathcal{H} \leq \mathcal{G}_m}} H^1(\mathcal{H}, \Bbbk)$$

are injective. Now, for each $\mathcal{H} \leq \mathcal{G}$, $H^1(\mathcal{H}, \Bbbk) = \operatorname{Hom}(\mathcal{H}^{\mathrm{ab}}, \Bbbk) \cong \mathcal{H}^{\mathrm{ab}}$, where the last isomorphism holds by our choice of $\Bbbk$. Therefore, injectivity of the above map is equivalent to $\mathcal{G}_m^{\mathrm{ab}}$ being generated by the images of all $\mathcal{H} \leq \mathcal{G}_m$ with $\mathcal{H} \in \mathfrak{X}$. □

3. The Cohen-Macaulay property

3.1. Continuing with the notation of §2.1, we now turn to the question of when the invariant algebra $\Bbbk[L]^{\mathcal{G}}$ is Cohen-Macaulay. Our principal tool will be Proposition 1.4. We remark that the Cohen-Macaulay hypothesis of Proposition 1.4 simplifies slightly in the setting of multiplicative actions: it suffices to assume that $\Bbbk[L]^{\mathcal{G}}$ is Cohen-Macaulay. Indeed, in this case the base ring $\Bbbk$ is also Cohen-Macaulay, because $\Bbbk[L]^{\mathcal{G}}$ is free over $\Bbbk$, and then $\Bbbk[L]$ is Cohen-Macaulay as well; see [BH, Exercise 2.1.23 and Theorems 2.1.9, 2.1.3(b)].

3.2. **Base rings.** Our main interest is in the case where $\Bbbk = \mathbb{Z}$. As the following lemma shows, if $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay, then so is $\Bbbk[L]^{\mathcal{G}}$ for any Cohen-Macaulay base ring $\Bbbk$.

Lemma. *The following are equivalent:*

(a) $\mathbb{Z}[L]^{\mathcal{G}}$ *is Cohen-Macaulay;*
(b) $\Bbbk[L]^{\mathcal{G}}$ *is Cohen-Macaulay whenever* $\Bbbk$ *is;*
(c) $\Bbbk[L]^{\mathcal{G}}$ *is Cohen-Macaulay for* $\Bbbk = \mathbb{Z}/(|\mathcal{G}|)$*;*
(d) $\mathbb{F}_p[L]^{\mathcal{G}}$ *is Cohen-Macaulay for all primes* p *dividing* $|\mathcal{G}|$.

Proof. (a) $\Rightarrow$ (b): Put $S = \Bbbk[L]^{\mathcal{G}}$ and consider the extension of rings $\Bbbk \hookrightarrow S$. This extension is free; see §2.1. By [BH, Exercise 2.1.23], S is Cohen-Macaulay if (and only if) $\Bbbk$ is Cohen-Macaulay and, for all $\mathfrak{P} \in \operatorname{Spec} S$, the fibre $S_{\mathfrak{P}}/\mathfrak{p}S_{\mathfrak{P}}$ is Cohen-Macaulay, where $\mathfrak{p} = \mathfrak{P} \cap \Bbbk$. But $S_{\mathfrak{P}}/\mathfrak{p}S_{\mathfrak{P}}$ is a localization of $(S/\mathfrak{p}S)_{\mathfrak{p}\setminus 0} \cong Q(\Bbbk/\mathfrak{p})[L]^{\mathcal{G}}$; see equation (2.1). Therefore, by [BH, Theorem 2.1.3(b)], it suffices to show that $Q(\Bbbk/\mathfrak{p})[L]^{\mathcal{G}}$ is Cohen-Macaulay. In other words, we may assume that $\Bbbk$ is a field. By [BH, Theorem 2.1.10], we may further assume that $\Bbbk = \mathbb{Q}$ or $\Bbbk = \mathbb{F}_p$. But equation (2.1) implies that $\mathbb{Q}[L]^{\mathcal{G}} = \mathbb{Z}[L]^{\mathcal{G}}_{\mathbb{Z}\setminus 0}$ and $\mathbb{F}_p[L]^{\mathcal{G}} \cong \mathbb{Z}[L]^{\mathcal{G}}/(p)$. Since $\mathbb{Z}[L]^{\mathcal{G}}$ is assumed Cohen-Macaulay, [BH, Theorem 2.1.3] implies that $\mathbb{Q}[L]^{\mathcal{G}}$ and $\mathbb{F}_p[L]^{\mathcal{G}}$ are Cohen-Macaulay, as desired.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (d): Write $|\mathcal{G}| = \prod_p p^{n_p}$. Then $\Bbbk[L] \cong \prod_p \mathbb{Z}/(p^{n_p})[L]^{\mathcal{G}}$ and $\mathbb{Z}/(p^{n_p})[L]^{\mathcal{G}}$ is a localization of $\Bbbk[L]^{\mathcal{G}}$. Therefore, $\mathbb{Z}/(p^{n_p})[L]^{\mathcal{G}}$ is Cohen-Macaulay, by [BH, Theorem 2.1.3(b)]. If $n_p \neq 0$, then it follows from [BH, Theorem 2.1.3(a)] that $\mathbb{Z}_{(p)}[L]^{\mathcal{G}}$ and $\mathbb{F}_p[L]^{\mathcal{G}} \cong \mathbb{Z}_{(p)}[L]^{\mathcal{G}}/(p)$ are Cohen-Macaulay.

(d) $\Rightarrow$ (a): First, (d) implies that $\mathbb{F}_p[L]^{\mathcal{G}}$ is Cohen-Macaulay for all primes p. For, if p does not divide $|\mathcal{G}|$, then $\mathbb{F}_p[L]^{\mathcal{G}}$ is always Cohen-Macaulay; see [BH, Corollary 6.4.6]. Now let $\mathfrak{P}$ be a maximal ideal of $\mathbb{Z}[L]$. Then $\mathfrak{P}\cap\mathbb{Z} = (p)$ for some prime p and $\mathbb{Z}[L]^{\mathcal{G}}_{\mathfrak{P}}/(p)$ is a localization of $\mathbb{Z}[L]^{\mathcal{G}}/(p) = \mathbb{F}_p[L]^{\mathcal{G}}$. Thus, $\mathbb{Z}[L]^{\mathcal{G}}_{\mathfrak{P}}/(p)$ is Cohen-Macaulay and [BH, Theorem 2.1.3(a)] further implies that $\mathbb{Z}[L]^{\mathcal{G}}_{\mathfrak{P}}$ is Cohen-Macaulay. Since, $\mathfrak{P}$ was arbitrary, (a) is proved. □

Since normal rings of (Krull) dimension at most 2 are Cohen-Macaulay, the implication (d) $\Rightarrow$ (b) of the lemma shows that $\Bbbk[L]^{\mathcal{G}}$ is certainly Cohen-Macaulay whenever $\Bbbk$ is Cohen-Macaulay and L has rank at most 2.

3.3. **Proof of the Theorem.** We are now ready to prove the Theorem stated in the Introduction. Recall that, for any subgroup $\mathcal{H} \leq \mathcal{G}$, $\mathcal{H}^{(2)}$ denotes the subgroup generated by the elements of $\mathcal{H}$ that act as bireflections on L or, equivalently, by the subgroups of $\mathcal{H}$ that belong to $\mathfrak{X}_2$; see (2.2). Throughout, we assume that $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay.

We first show that $\mathcal{G}_m/\mathcal{G}_m^{(2)}$ is a perfect group for all $m \in L$. Put $\Bbbk = \mathbb{Z}/(|\mathcal{G}|)$. Then $\Bbbk[L]^{\mathcal{G}}$ is Cohen-Macaulay, by Lemma 3.2. Therefore, the restriction

$$H^1(\mathcal{G}, \Bbbk[L]) \to \prod_{\mathcal{H}\in\mathfrak{X}_2} H^1(\mathcal{H}, \Bbbk[L])$$

is injective, by Proposition 1.4; see the remark in §3.1. Corollary 2.5 yields that all $\mathcal{G}_m^{\mathrm{ab}}$ are generated by the images of the subgroups $\mathcal{H} \leq \mathcal{G}_m$ with $\mathcal{H} \in \mathfrak{X}_2$. In other words, each $\mathcal{G}_m^{\mathrm{ab}}$ is generated by the images of the bireflections in $\mathcal{G}_m$. Therefore, $\left(\mathcal{G}_m/\mathcal{G}_m^{(2)}\right)^{\mathrm{ab}} = 1$, as desired.

Now assume that $\mathcal{G}$ acts non-trivially on L. Our goal is to show that some isotropy group $\mathcal{G}_m$ is non-perfect. Suppose otherwise. Replacing $\mathcal{G}$ by $\mathcal{G}/\operatorname{Ker}_{\mathcal{G}}(L)$ we may assume that $1 \neq \mathcal{G}$ acts faithfully on L. Then $\mathfrak{X}_k = \{1\}$ for all $k < 8$, by Proposition 2.4. It follows that

$$k = \inf\{i > 0 \mid H^i(\mathcal{G}, \Bbbk[L]) \neq 0\} \geq 7 \ .$$

Indeed, if $k < 7$, then Proposition 1.4 implies that $0 \neq H^k(\mathcal{G}, \Bbbk[L])$ embeds into $\prod_{\mathcal{H}\in\mathfrak{X}_{k+1}} H^k(\mathcal{H}, \Bbbk[L])$ which is trivial, because $\mathfrak{X}_{k+1} = \{1\}$. By Lemma 2.5 with

$\mathfrak{X} = \{1\}$, we conclude that

$$H^i(\mathcal{G}_m, \Bbbk) = 0 \text{ for all } m \in L \text{ and all } 0 < i < 7.$$

On the other hand, choosing $\mathcal{G}_m$ minimal with $\mathcal{G}_m \neq 1$, we know by Lemmas 2.3 and 2.4(b) that $\mathcal{G}_m$ is isomorphic to the binary icosahedral group $2.\mathcal{A}_5$. The cohomology of $2.\mathcal{A}_5$ is 4-periodic (see [Br, p. 155]). Hence, $H^3(\mathcal{G}_m, \Bbbk) \cong H^{-1}(\mathcal{G}_m, \Bbbk) = \operatorname{ann}_{\Bbbk}(\sum_{\mathcal{G}_m} g) \cong \mathbb{Z}/(|\mathcal{G}_m|) \neq 0$. This contradiction completes the proof of the Theorem. □

3.4. **Rational invariance.** We now show that the Cohen-Macaulay property of $\Bbbk[L]^{\mathcal{G}}$ depends only on the rational isomorphism class of the $\mathcal{G}$-lattice L. Recall that $\mathcal{G}$-lattices L and L' are said to be *rationally isomorphic* if $L \otimes_{\mathbb{Z}} \mathbb{Q} \cong L' \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}[\mathcal{G}]$-modules. In this section, $\Bbbk$ denotes any commutative base ring.

Proposition. *If $\Bbbk[L]^{\mathcal{G}}$ is Cohen-Macaulay, then so is $\Bbbk[L']^{\mathcal{G}}$ for any $\mathcal{G}$-lattice L' that is rationally isomorphic to L.*

Proof. Assume that $L \otimes_{\mathbb{Z}} \mathbb{Q} \cong L' \otimes_{\mathbb{Z}} \mathbb{Q}$. Replacing L' by an isomorphic copy inside $L \otimes_{\mathbb{Z}} \mathbb{Q}$, we may assume that $L \supseteq L'$ and L/L' is finite. Then $\Bbbk[L]$ is finite over $\Bbbk[L']$ which in turn is integral over $\Bbbk[L']^{\mathcal{G}}$. Therefore, $\Bbbk[L]$ is integral over $\Bbbk[L']^{\mathcal{G}}$, and hence so is $\Bbbk[L]^{\mathcal{G}}$.

We now invoke a ring-theoretic result of Hochster and Eagon [HE] (or see [BH, Theorem 6.4.5]): Let $R \supseteq S$ be an integral extension of commutative rings having a Reynolds operator, that is, an S-linear map $R \to S$ that restricts to the identity on S. If R is Cohen-Macaulay, then S is Cohen-Macaulay as well.

To construct the requisite Reynolds operator, consider the truncation map

$$\pi \colon \Bbbk[L] \to \Bbbk[L'] \ , \quad \sum_{m \in L} k_m \mathbf{x}^m \mapsto \sum_{m \in L'} k_m \mathbf{x}^m \ .$$

This is a Reynolds operator for the extension $\Bbbk[L] \supseteq \Bbbk[L']$ that satisfies $\pi(g(f)) = g(\pi(f))$ for all $g \in \mathcal{G}$, $f \in \Bbbk[L]$. Therefore, π restricts to a Reynolds operator $\Bbbk[L]^{\mathcal{G}} \to \Bbbk[L']^{\mathcal{G}}$ and the proposition follows. □

The proposition in particular allows us to reduce the general case of the Cohen-Macaulay problem for multiplicative invariants to the case of effective $\mathcal{G}$-lattices. Recall that the $\mathcal{G}$-lattice L is *effective* if $L^{\mathcal{G}} = 0$. For any $\mathcal{G}$-lattice L, the quotient $L/L^{\mathcal{G}}$ is an effective $\mathcal{G}$-lattice; this follows, for example, from the fact that L is rationally isomorphic to the $\mathcal{G}$-lattice $L^{\mathcal{G}} \oplus L/L^{\mathcal{G}}$.

Corollary. *$\Bbbk[L]^{\mathcal{G}}$ is Cohen-Macaulay if and only if this holds for $\Bbbk[L/L^{\mathcal{G}}]^{\mathcal{G}}$.*

Proof. By the proposition, we may replace L by $L' = L^{\mathcal{G}} \oplus L/L^{\mathcal{G}}$. But $\Bbbk[L']^G \cong \Bbbk[L/L^{\mathcal{G}}]^{\mathcal{G}} \otimes_{\Bbbk} \Bbbk[L^{\mathcal{G}}]$, a Laurent polynomial algebra over $k[L/L^{\mathcal{G}}]^{\mathcal{G}}$. Thus, by [BH, Theorems 2.1.3 and 2.1.9], $\Bbbk[L']^{\mathcal{G}}$ is Cohen-Macaulay if and only if $\Bbbk[L/L^{\mathcal{G}}]^{\mathcal{G}}$ is Cohen-Macaulay. The corollary follows. □

3.5. **Remarks and examples.**

3.5.1. *Abelian bireflection groups.* It is not hard to show that if $\mathcal{G}$ is a finite abelian group acting as a bireflection group on the lattice L, then $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay. Using Corollary 3.4 and an induction on $\operatorname{rank} L$, the proof reduces to the verification that $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay for $L = \mathbb{Z}^n$ and $\mathcal{G} = \mathrm{diag}(\pm 1, \ldots, \pm 1) \cap \mathrm{SL}_n(\mathbb{Z})$. Direct computation shows that, for $n \geq 2$,

$$\mathbb{Z}[L]^{\mathcal{G}} = \mathbb{Z}[\xi_1, \ldots, \xi_n] \oplus \eta \mathbb{Z}[\xi_1, \ldots, \xi_n],$$

where $\xi_i = \mathbf{x}^{e_i} + \mathbf{x}^{-e_i}$ is the $\mathcal{G}$-orbit sum of the standard basis element $e_i \in \mathbb{Z}^n$ and η is the orbit sum of $\sum_i e_i = (1, \ldots, 1)$.

It would be worthwhile to try and extend this result to larger classes of bireflection groups. The aforementioned classification of bireflection groups in [GuS] will presumably be helpful in this endeavor.

3.5.2. *Subgroups of reflection groups.* Assume that $\mathcal{G}$ acts as a reflection group on the lattice L and let $\mathcal{H}$ be a subgroup of $\mathcal{G}$ with $[\mathcal{G} : \mathcal{H}] = 2$. Then $\mathcal{H}$ acts as a bireflection group. (More generally, if $\mathcal{G}$ acts as a k-reflection group and $[\mathcal{G} : \mathcal{H}] = m$, then $\mathcal{H}$ acts as a km-reflection group; see [L_1].) Presumably $\mathbb{Z}[L]^{\mathcal{H}}$ will always be Cohen-Macaulay, but I have no proof. For an explicit example, let $\mathcal{G} = \mathcal{S}_n$ be the symmetric group on $\{1, \ldots, n\}$ and let $L = U_n$ be the standard permutation lattice for $\mathcal{S}_n$; so $U_n = \bigoplus_{i=1}^n \mathbb{Z}e_i$ with $s(e_i) = e_{s(i)}$ for $s \in \mathcal{S}_n$. Transpositions act as reflections on U_n and 3-cycles as bireflections. Let $\mathcal{A}_n \leq \mathcal{S}_n$ denote the alternating group. To compute $\mathbb{Z}[U_n]^{\mathcal{A}_n}$, put $x_i = \mathbf{x}^{e_i} \in \mathbb{Z}[U_n]$. Then $\mathbb{Z}[U_n] = \mathbb{Z}[x_1, \ldots, x_n][s_n^{-1}]$, where $s_n = \mathbf{x}^{\sum_1^n e_i} = \prod_1^n x_i$ is the n^{th} elementary symmetric function, and $\mathcal{S}_n$ acts via $s(x_i) = x_{s(i)}$ $(s \in \mathcal{S}_n)$. Therefore, $\mathbb{Z}[U_n]^{\mathcal{A}_n} = \mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{A}_n}[s_n^{-1}]$. The ring $\mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{A}_n}$ of polynomial $\mathcal{A}_n$-invariants has the following form; see [S, Theorem 1.3.5]: $\mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{A}_n} = \mathbb{Z}[s_1, \ldots, s_n] \oplus d\mathbb{Z}[s_1, \ldots, s_n]$, where s_i is the i^{th} elementary symmetric function and

$$d = \tfrac{1}{2}\left(\Delta + \Delta_+\right)$$

with $\Delta_+ = \prod_{i<j}(x_i + x_j)$ and $\Delta = \prod_{i<j}(x_i - x_j)$, the Vandermonde determinant. Thus,

$$\mathbb{Z}[U_n]^{\mathcal{A}_n} = \mathbb{Z}[s_1, \ldots, s_{n-1}, s_n^{\pm 1}] \oplus d\mathbb{Z}[s_1, \ldots, s_{n-1}, s_n^{\pm 1}].$$

This is Cohen-Macaulay, being free over $\mathbb{Z}[s_1, \ldots, s_{n-1}, s_n^{\pm 1}]$.

3.5.3. *$\mathcal{S}_n$-lattices.* If L is a lattice for the symmetric group $\mathcal{S}_n$ such that $\mathbb{Z}[L]^{\mathcal{S}_n}$ is Cohen-Macaulay, then the Theorem implies that $\mathcal{S}_n$ acts as a bireflection group on L, and hence on all simple constituents of $L \otimes_{\mathbb{Z}} \mathbb{Q}$. The simple $\mathbb{Q}[\mathcal{S}_n]$-modules are the Specht modules S^λ for partitions λ of n. If $n \geq 7$, then the only partitions λ so that $\mathcal{S}_n$ acts as a bireflection group on S^λ are (n), (1^n) and $(n-1, 1)$; this follows from the lists in [Hu] and [W]. The corresponding Specht modules are trivial module, $\mathbb{Q}$, the sign module $\mathbb{Q}^-$, and the rational root module $A_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q}$, where $A_{n-1} = \{\sum_i z_i e_i \in U_n \mid \sum_i z_i = 0\}$ and U_n is as in §3.5.2. Thus, if $n \geq 7$ and $\mathbb{Z}[L]^{\mathcal{S}_n}$ is Cohen-Macaulay, then we must have

$$L \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^r \oplus \left(\mathbb{Q}^-\right)^s \oplus \left(A_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q}\right)^t$$

with $s + t \leq 2$. In most cases, $\mathbb{Z}[L]^{\mathcal{S}_n}$ is easily seen to be Cohen-Macaulay. Indeed, we may assume $r = 0$ by Corollary 3.4. If $s + t \leq 1$, then $\mathcal{S}_n$ acts as a reflection group on L and so $\mathbb{Z}[L]^{\mathcal{S}_n}$ is Cohen-Macaulay by [L_2]. For $t = 0$ we may quote the last remark in §3.2. This leaves the cases $s = t = 1$ and $s = 0$, $t = 2$ to consider.

If $s = t = 1$, then add a copy of $\mathbb{Q}$ so that L is rationally isomorphic to $U_n \oplus \mathbb{Z}^-$. Using the notation of §3.5.2 and putting $t = \mathbf{x}^{(0_{U_n},1)} \in \mathbb{Z}[U_n \oplus \mathbb{Z}^-]$ the invariants are

$$\mathbb{Z}[U_n \oplus \mathbb{Z}^-]^{\mathcal{S}_n} = R \oplus R\varphi$$

with $R = \mathbb{Z}[s_1, \dots, s_{n-1}, s_n^{\pm 1}, t + t^{-1}]$ and $\varphi = \frac{1}{2}(\Delta_+ + \Delta)t + \frac{1}{2}(\Delta_+ - \Delta)t^{-1}$.

If $s = 0$ and $t = 2$, then we may replace L by the lattice $U_n^2 = U_n \oplus U_n$. By Lemma 3.2 $\mathbb{Z}[U_n^2]^{\mathcal{S}_n}$ is Cohen-Macaulay precisely if $\mathbb{F}_p[U_n^2]^{\mathcal{S}_n}$ is Cohen-Macaulay for all primes $p \leq n$. As in §3.5.2, one sees that $\mathbb{F}_p[U_n^2]^{\mathcal{S}_n}$ is a localization of the algebra "vector invariants" $\mathbb{F}_p[x_1, \dots, x_n, y_1, \dots, y_n]^{\mathcal{S}_n}$. By [$K_2$, Corollary 3.5], this algebra is known to be Cohen-Macaulay for $n/2 < p \leq n$, but the primes $p \leq n/2$ apparently remain to be dealt with.

3.5.4. *Ranks* ≤ 4. As was pointed out in §3.2, $\mathbb{Z}[L]^{\mathcal{G}}$ is always Cohen-Macaulay when $\operatorname{rank} L \leq 2$.

For $L = \mathbb{Z}^3$, there are 32 $\mathbb{Q}$-classes of finite subgroups $\mathcal{G} \leq \mathrm{GL}_3(\mathbb{Z})$. All $\mathcal{G}$ are solvable; in fact, their orders divide 48. The Sylow 3-subgroup $\mathcal{H} \leq \mathcal{G}$, if non-trivial, is generated by a bireflection of order 3. Thus, $\mathbb{F}_3[L]^{\mathcal{H}}$ is Cohen-Macaulay, and hence so is $\mathbb{F}_3[L]^{\mathcal{G}}$. Therefore, by Lemma 3.2, $\mathbb{Z}[L]^{\mathcal{G}}$ is Cohen-Macaulay if and only if $\mathbb{F}_2[L]^{\mathcal{G}}$ is Cohen-Macaulay, and for this to occur, $\mathcal{G}$ must be generated by bireflections. It turns out that 3 of the 32 $\mathbb{Q}$-classes consist of non-bireflection groups; these classes are represented by the cyclic groups

$$\left\langle \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \right\rangle , \quad \left\langle \begin{pmatrix} & 1 & \\ -1 & & \\ & & -1 \end{pmatrix} \right\rangle , \quad \left\langle \begin{pmatrix} & & -1 \\ -1 & & \\ & -1 & \end{pmatrix} \right\rangle$$

of orders 2, 4 and 6 (the latter two classes each split into 2 $\mathbb{Z}$-classes). For the $\mathbb{Q}$-classes consisting of bireflection groups, Pathak [Pk] has checked explicitly that $\mathbb{F}_2[L]^{\mathcal{G}}$ is indeed Cohen-Macaulay.

In rank 4, there are 227 $\mathbb{Q}$-classes of finite subgroups $\mathcal{G} \leq \mathrm{GL}_4(\mathbb{Z})$. All but 5 of them consist of solvable groups and 4 of the non-solvable classes are bireflection groups, the one exception being represented by $\mathcal{S}_5$ acting on the signed root lattice $\mathbb{Z}^- \otimes_{\mathbb{Z}} A_4$. Thus, if the group $\mathcal{G}/\mathcal{G}^{(2)}$ is perfect, then it is actually trivial, that is, $\mathcal{G}$ is a bireflection group. It also turns out that, in this case, all isotropy groups $\mathcal{G}_m$ are bireflection groups. There are exactly 71 $\mathbb{Q}$-classes that do not consist of bireflection groups. By the foregoing, they lead to non-Cohen-Macaulay multiplicative invariant algebras. The $\mathbb{Q}$-classes consisting of bireflection groups have not been systematically investigated yet. The searches in rank 4 were done with [GAP].

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